Interval valued intuitionistic (S,T)-fuzzy H_v-submodules

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Abstract. On the basis of the concept of the interval valued intuitionistic fuzzy sets introduced by K.Atanassov, the notion of interval valued intuitionistic fuzzy H_v-submodules of an H_v-module with respect to t-norm T and s-norm S is given and the characteristic properties are described. The homomorphic image and the inverse image are investigated.In particular, the connections between interval valued intuitionistic (S,T)-fuzzy H_v-submodules and interval valued intuitionistic (S,T)-fuzzy submodules are discussed.

Key words and phrases H_v-module. interval valued intuitionistic (S,T)-fuzzy H_v-submodule, interval valued intuitionistic (S,T)-fuzzy submodule.

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1 Introduction

The concept of hyperstructure was introduced in 1934 by Marty [8] at the 8th Congress of Scandinavian Mathematicians. Hypersructures have many applications to several branches of both pure and applied sciences (see for example [4] and [5]). Vougiouklis [12, 10] introduced a new class of hyperstructures called now H_v-structures, and Davvaz [7] surveyed the theory of H_v-structures. After the introduction of fuzzy sets by Zadeh [14], there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [1] is one among them. For more details on intuitionistic fuzzy sets, we refer the reader to [1, 2, 3]. In 1975, Zadeh [15] introduced the concept of interval valued fuzzy subsets, where the values of the membership functions are intervals of numbers instead of the numbers.

Such fuzzy sets have some applications in the technological scheme of the functioning of a silo-farm with pneumatic transportation in a plastic products company and in medicine (see the book [3]).

In this paper, we introduce the notion of interval valued intuitionistic (S,T)-fuzzy H_v-submodules of an H_v-module and describe the characteristic properties. We give the homomorphic image and the inverse image. In particular, we discuss the connections between interval valued intuitionistic (S,T)-fuzzy H_v-submodules and interval valued intuitionistic (S,T)-fuzzy submodules.

2 Preliminaries

In this section, we recall some basic definitions for the sake of completeness.

As it is well known [12], a *hyperstructure* is a non-empty set H together with a map $\cdot : H \times H \to P^*(H)$, called a *hyperoperation*, where $P^*(H)$ is the family of all non-empty subsets of H. The image of pair (x, y) is denoted by $x \cdot y$. If $x \in H$ and $A, B \subseteq H$, then by $A \cdot B$, $A \cdot x$ and $x \cdot B$ we mean

$$A \cdot B = \bigcup_{a \in A, b \in B} a \cdot b, \quad A \cdot x = A \cdot \{x\} \ \text{ and } \ x \cdot B = \{x\} \cdot B,$$

respectively.

Definition 2.1. A hyperstructure $(H, \cdot)$ is called an H_v*-semigroup* if

$$(x \cdot (y \cdot z)) \cap ((x \cdot y) \cdot z) \neq \emptyset \ \text{ for all } x, y, z \in H.$$

An H_v-semigroup in which $a \cdot H = H \cdot a = H$ is valid for every $a \in H$ is called an H_v*-group.*

The last condition means that for any $a, h \in H$ there exist $u, v \in H$ such that $h \in a \cdot u$ and $h \in v \cdot a$. An H_v-group $(H, \cdot)$ satisfying for all $x, y \in H$ the condition $x \cdot y \cap y \cdot x \neq \emptyset$ is called *weak commutative.*

Definition 2.2. An H_v*-ring* is a system $(R, +, \cdot)$ with two hyperoperations satisfying the following axioms:
(i) $(R, +)$ is an H_v-group;,
(ii) $(R, \cdot)$ is an H_v-semigroup,
(iii) the multiplication $\cdot$ is weak distributive with respect to the addition $+$, i.e.,

$$(x \cdot (y + z)) \cap (x \cdot y + x \cdot z) \neq \emptyset,$$
$$((x + y) \cdot z) \cap (x \cdot z + y \cdot z) \neq \emptyset$$

for all $x, y, z \in R$.

Definition 2.3 ([11]). A non-empty set M is an H_v*-module over an* H_v*-ring* R if $(M, +)$ is a weak commutative H_v-group and there exists the map $\cdot : R \times M \to P^*(M)$, $(r, x) \mapsto r \cdot x$, such that for all $a \in R$ and $x, y \in M$, we have
(i) $(a \cdot (x + y)) \cap (a \cdot x + a \cdot y) \neq \emptyset$,
(ii) $((a + b) \cdot x) \cap (a \cdot x + b \cdot x) \neq \emptyset$,
(iii) $((a \cdot b) \cdot x) \cap (a \cdot (b \cdot x)) \neq \emptyset$.

A non-empty subset S of M is an H_v*-submodule* of M if $(S, +)$ is an H_v-subgroup of $(M, +)$ and $R \cdot S \subseteq S$.

It is clear that an arbitrary ring (module) will be an H_v-ring (H_v-module) if we identify x with $\{x\}$. Others interesting examples are given below.

Example 2.4. Let $(M, +, \cdot)$ be an ordinary module over a ring R with a center $Z(R)$. On $R \times M$ we can define three hyperoperations P^*, P_+ and P^*_+ putting for all $(r, x) \in R \times M$:

(1) $rP^*x = (rP)x$ if $P \subseteq R$,

(2) $rP_+x = r(P + x)$ if $P \subseteq M$,

(3) $rP^*_+x = (rP_1)(P_2 + x)$ if $P_1 \subseteq R$ and $P_2 \subseteq M$.

Then, as it is not difficult to verify,

(a) $(M, +, P^*)$ is an H_v-module over R, if there exists $p \in P \cap Z(R)$ such that $p^2 \in P$,

(b) $(M, +, P_+)$ is an H_v-module over R, if the zero 0 of $(M, +)$ belongs to $P \subseteq M$,

(c) $(M, +, P_+^*)$ is an H_v-module over R, if there exist $p_1 \in P_1 \cap Z(R)$ such that $p_1^2 = p_1$ and $p_2 \in P_2 \subseteq M$ such that $p_1 \cdot p_2 = 0$.

According to Zadeh [14], a *fuzzy set* μ_F defined on a non-empty set X, i.e. a map $\mu_F : X \to [0, 1]$, can be identified with the set $F = \{(x, \mu_F(x)) \mid x \in X\}$.

Definition 2.5 ([7]). A fuzzy set F of an H_v-module M over an H_v-ring R is said to be a *fuzzy H_v-submodule* of M if:

(i) $\min\{\mu_F(x), \mu_F(y)\} \le \inf\limits_{\alpha \in x+y} \mu_F(\alpha)$ for all $x, y \in M$,

(ii) for all $x, a \in M$, there exists $y \in M$ such that $x \in a + y$ and $\min\{\mu_F(a), \mu_F(x)\} \le \mu_F(y)$,

(iii) for all $x, a \in M$, there exists $z \in M$ such that $x \in z + a$ and $\min\{\mu_F(a), \mu_F(x)\} \le \mu_F(z)$,

(iv) $\mu_F(x) \le \inf\limits_{\alpha \in r \cdot x} \mu_F(\alpha)$ for all $r \in R$ and $x \in M$.

By an *interval number* $\tilde{a}$ we mean (cf. [2]) an interval $[a^-, a^+]$, where $0 \le a^- \le a^+ \le 1$. The set of all interval numbers is denoted by $D[0, 1]$. The interval $[a, a]$ is identified with the number $a \in [0, 1]$.

For interval numbers $\widetilde{a}_i = [a_i^-, a_i^+] \in D[0, 1]$, $i \in I$, we define

$$\inf \widetilde{a}_i = [\bigwedge_{i \in I} a_i^-, \bigwedge_{i \in I} a_i^+], \qquad \sup \widetilde{a}_i = [\bigvee_{i \in I} a_i^-, \bigvee_{i \in I} a_i^+]$$

and put

(1) $\widetilde{a}_1 \le \widetilde{a}_2 \Longleftrightarrow a_1^- \le a_2^-$ and $a_1^+ \le a_2^+$,

(2) $\widetilde{a}_1 = \widetilde{a}_2 \Longleftrightarrow a_1^- = a_2^-$ and $a_1^+ = a_2^+$,

(3) $\widetilde{a}_1 < \widetilde{a}_2 \Longleftrightarrow \widetilde{a}_1 \le \widetilde{a}_2$ and $\widetilde{a}_1 \ne \widetilde{a}_2$,

(4) $k\widetilde{a} = [ka^-, ka^+]$, whenever $0 \le k \le 1$.

In is clear that $(D[0, 1], \le, \vee, \wedge)$ is a complete lattice with $0 = [0, 0]$ as the least element and $1 = [1, 1]$ as the greatest element.

By an *interval valued fuzzy set* F on X we mean (sf. [15]) the set

$$F = \{(x, [\mu_F^-(x), \mu_F^+(x)]) \mid x \in X\},$$

where μ_F^- and μ_F^+ are two fuzzy subsets of X such that $\mu_F^-(x) \le \mu_F^+(x)$ for all $x \in X$. Putting $\mu_F(x) = [\mu_F^-(x), \mu_F^+(x)]$, we see that $F = \{(x, \mu_F(x)) \mid x \in X\}$, where $\mu_F : X \to D[0, 1]$.

As it is well-known, any function $\delta : [0, 1] \times [0, 1] \to [0, 1]$ such that $\delta(x, y) = \delta(y, x)$, $\delta(x, x) = x$, $\delta(\delta(x, y), z) = \delta(x, \delta(y, z))$ and $\delta(x, u) \le \delta(x, w)$ for all $x, y, z, u, w \in [0, 1]$, where $u \le w$ is called an *idempotent t-norm* if $\delta(x, 1) = x$, and an *idempotent s-norm* if $\delta(1, 1) = 1$ and $\delta(x, 0) = x$ for all $x \in [0, 1]$.

If δ is an idempotent t-norm (s-norm), then the mapping $\Delta : D[0,1] \times D[0,1] \to D[0,1]$ defined by $\Delta(\widetilde{a}_1, \widetilde{a}_2) = [\delta(a_1^-, a_2^-), \delta(a_1^+, a_2^+)]$ is, as it is not difficult to verify, an idempotent t-norm (s-norm, respectively) and is called an *idempotent interval t-norm* (*s-norm*, respectively).

According to Atanassov (cf. [1, 3]) an *interval valued intuitionistic fuzzy set* on X is defined as the object of the form

$$A = \{(x, \widetilde{M}_A(x), \widetilde{N}_A(x)) \mid x \in X\},$$

where $\widetilde{M}_A(x)$ and $\widetilde{N}_A(x)$ are interval valued fuzzy sets on X such that

$$0 \leq \sup \widetilde{M}_A(x) + \sup \widetilde{N}_A(x) \leq 1 \quad \text{for all } \ x \in X.$$

For the sake of simplicity, in the sequel such interval valued intuitionistic fuzzy set will be denoted by $A = (\widetilde{M}_A, \widetilde{N}_A)$.

3 Interval valued intuitionistic (S, T)-fuzzy H_v-submodules

In what follow, let M denote an H_v-module over an H_v-ring R unless otherwise specified.

Definition 3.1. Let T (resp. S) be an idempotent interval t-norm (resp. s-norm). An interval valued intuitionistic fuzzy set $A = (\widetilde{M}_A, \widetilde{N}_A)$ of M is called an *interval valued intuitionistic (S, T)-fuzzy H_v-submodule* of M if the following condition hold:

(1) $T(\widetilde{M}_A(x), \widetilde{M}_A(y)) \leq \inf\limits_{\alpha \in x+y} \widetilde{M}_A(\alpha)$ and $S(\widetilde{N}_A(x), \widetilde{N}_A(y)) \geq \sup\limits_{\alpha \in x+y} \widetilde{N}_A(\alpha)$, $\forall x, y \in M$,

(2) for all $x, a \in M$ there exists $y \in M$ such that $x \in a + y$, $T(\widetilde{M}_A(x), \widetilde{M}_A(a)) \leq \widetilde{M}_A(y)$ and $S(\widetilde{N}_A(x), \widetilde{N}_A(a)) \geq \widetilde{N}_A(y)$,

(3) for all $x, a \in M$ there exists $z \in M$ such that $x \in z + a$, $T(\widetilde{M}_A(x), \widetilde{M}_A(a)) \leq \widetilde{M}_A(z)$ and $S(\widetilde{N}_A(x), \widetilde{N}_A(a)) \geq \widetilde{N}_A(z)$,

(4) $\widetilde{M}_A(x) \leq \inf\limits_{\alpha \in r \cdot x} \widetilde{M}_A(\alpha)$ and $\widetilde{N}_A(x) \geq \sup\limits_{\alpha \in r \cdot x} \hat{N}_A(\alpha)$ for all $x \in M$ and $r \in R$.

With any interval valued intuitionistic fuzzy set $A = (\widetilde{M}_A, \widetilde{N}_A)$ of M are connected two levels:

$$U(\widetilde{M}_A; [t, s]) = \{x \in X \mid \widetilde{M}_A(x) \geq [t, s]\},$$

and

$$L(\widetilde{N}_A; [t, s]) = \{x \in X \mid \widetilde{N}_A(x) \leq [t, s]\}.$$

Theorem 3.2. *Let T (resp. S) be an idempotent interval t-norm (resp. s-norm), then $A = (\widetilde{M}_A, \widetilde{N}_A)$ is an interval valued intuitionistic (S, T)-fuzzy H_v-submodule of M if and only if every for all $t, s \in [0, 1]$, $t \leq s$, $U(\widetilde{M}_A; [t, s])$ and $L(\widetilde{N}_A; [t, s])$ are H_v-submodules of M.*

Proof. Let $A = (\widetilde{M}_A, \widetilde{N}_A)$ be an interval valued intuitionistic (S, T)-fuzzy H_v-submodule of M. Then for every $x, y \in U(\widetilde{M}_A; [t, s])$ we have $\widetilde{M}_A(x) \geq [t, s]$ and $\widetilde{M}_A(y) \geq [t, s]$. Hence $T(\widetilde{M}_A(x), \widetilde{M}_A(y)) \geq T([t, s], [t, s]) = [t, s]$, and so $\inf\limits_{\alpha \in x+y} \widetilde{M}_A(\alpha) \geq [t, s]$. Therefore $\alpha \in U(\widetilde{M}_A; [t, s])$ for every $\alpha \in x + y$, so $x + y \subseteq U(\widetilde{M}_A; [t, s])$. Thus, for every $a \in U(\widetilde{M}_A; [t, s])$,

we have $a + U(\widetilde{M}_A; [t, s]) \subseteq U(\widetilde{M}_A; [t, s])$. On the other hand, for $x, a \in U(\widetilde{M}_A; [t, s])$ there exists $y \in H$ such that $x \in a+y$ and $T(\widetilde{M}_A(x), \widetilde{M}_A(a)) \le \widetilde{M}_A(y)$. But $T(\widetilde{M}_A(x), \widetilde{M}_A(a)) \ge [t, s]$ for all $x, a \in U(\widetilde{M}_A; [t, s])$, so $\widetilde{M}_A(y) \ge [t, s]$, that is, $y \in U(\widetilde{M}_A; [t, s])$. Whence $U(\widetilde{M}_A; [t, s]) \subseteq a + U(\widetilde{M}_A; [t, s])$, and, in the consequence $U(\widetilde{M}_A; [t, s]) = a + U(\widetilde{M}_A; [t, s])$. Similarly, we can prove that $U(\widetilde{M}_A; [t, s]) = U(\widetilde{M}_A; [t, s]) + a$. That is, $a + U(\widetilde{M}_A; [t, s]) = U(\widetilde{M}_A; [t, s]) = U(\widetilde{M}_A; [t, s]) + a$. This proves that $(U(\widetilde{M}_A; [t, s]), +)$ is an H_v-subgroup of $(M, +)$.

If $r \in R$ and $x \in U(\widetilde{M}_A; [t, s])$, then $\widetilde{M}_A(x) \ge [t, s]$, which means that $\inf\limits_{\alpha \in r \cdot x} \widetilde{M}_A(\alpha) \ge [t, s]$. So, $\alpha \in U(\widetilde{M}_A; [t, s])$ for every $\alpha \in r \cdot x$. Therefore, $r \cdot x \subseteq U(\widetilde{M}_A; [t, s])$, i.e. $r \cdot U(\widetilde{M}_A; [t, s]) \subseteq U(\widetilde{M}_A; [t, s])$. This proves that $U(\widetilde{M}_A; [t, s])$ is an H_v-submodule of M. Similarly, we can show that $L(\widetilde{N}_A; [t, s])$ is an H_v-submodule of M.

Conversely, assume that for every $[t, s] \in D[0, 1]$ any non-empty $U(\widetilde{M}_A; [t, s])$ is an H_v-submodule of M. If $[t_0, s_0] = T(\widetilde{M}_A(x), \widetilde{M}_A(y))$ for some $x, y \in H$, then $x, y \in U(\widetilde{M}_A; [t_0, s_0])$, and so $x + y \subseteq U(\widetilde{M}_A; [t_0, s_0])$. Therefore $\alpha \in U(\widetilde{M}_A; [t_0, s_0])$ for every $\alpha \in x + y$, and so $\inf\limits_{\alpha \in x+y} \widetilde{M}_A(\alpha) \ge T(\widetilde{M}_A(x), \widetilde{M}_A(y))$. Now, if $[t_1, s_1] = T(\widetilde{M}_A(a), \widetilde{M}_A(x))$ for some $a, x \in H$, then $a + x \in U(\widetilde{M}_A; [t_1, s_1])$, so there exists $y \in U(\widetilde{M}_A; [t_1, s_1])$ such that $x \in a + y$. But for $y \in U(\widetilde{M}_A; [t_1, s_1])$ we have $\widetilde{M}_A(y) \ge [t_1, s_1]$, whence $\widetilde{M}_A(y) \ge T(\widetilde{M}_A(a), \widetilde{M}_A(x))$. Similarly, we can show that for $a, x \in H$ there exists $z \in H$ such that $x \in z + a$ and $\widetilde{M}_A(z) \ge T(\widetilde{M}_A(a), \widetilde{M}_A(x))$. If $[t_2, s_2] = \widetilde{M}_A(x)$ for some $x \in M$, then $x \in U(\widetilde{M}_A; [t_2, s_2])$, and so $r \cdot x \subseteq U(\widetilde{M}_A; [t_2, s_2])$ for every $r \in R$. Therefore for every $\alpha \in r \cdot x$, we have $\alpha \in U(\widetilde{M}_A; [t_2, s_2])$, consequently $\inf\limits_{\alpha \in r \cdot x} \widetilde{M}_A(\alpha) \ge [t_2, s_2] = \widetilde{M}_A(x)$. This proves that $\widetilde{M}_A$ is an interval valued T-fuzzy H_v-submodule of M.

Similarly, we can show that $\widetilde{N}_A$ is an interval valued S-fuzzy H_v-submodule of M. Therefore, $A = (\widetilde{M}_A, \widetilde{N}_A)$ is an interval valued intuitionistic (S, T)-fuzzy H_v-submodule of M. $\square$

Definition 3.3. Let $f : X \to Y$ be a mapping and $A = (\widetilde{M}_A, \widetilde{N}_A)$, $B = (\widetilde{M}_B, \widetilde{N}_B)$ an interval valued intuitionistic sets X and Y, respectively. Then the *image* $f[A] = (f(\widetilde{M}_A), f(\widetilde{N}_A))$ of A is the interval valued intuitionistic fuzzy set of Y defined by

$$f(\widetilde{M}_A)(y) = \begin{cases} \sup\limits_{z \in f^{-1}(y)} \widetilde{M}_A(z) & \text{if } f^{-1}(y) \neq \emptyset \\ [0, 0] & \text{otherwise} \end{cases}$$

$$f(\widetilde{N}_A)(y) = \begin{cases} \inf\limits_{z \in f^{-1}(y)} \widetilde{N}_A(z) & \text{if } f^{-1}(y) \neq \emptyset \\ [1, 1] & \text{otherwise} \end{cases}$$

for all $y \in Y$.

The inverse image $f^{-1}(B)$ of B is an interval valued intuitionistic fuzzy set defined by $f^{-1}(\widetilde{M}_B(x) = \widetilde{M}_B(f(x))$, $f^{-1}(\widetilde{N}_B)(x) = \widetilde{N}_B(f(x))$ for all $x \in X$.

Definition 3.4 ([3]). Let M and N be two H_v-modules over an H_v-ring R. A mapping $f : M \to N$ is called an *H_v-homomorphism* or *weak homomorphism* if for all $x, y \in M$ and $r \in R$, the following relations hold: $f(x+y) \bigcap (f(x) + f(y)) \neq \emptyset$ and $f(r \cdot x) \bigcap r \cdot f(x) \neq \emptyset$. f is called an *inclusion homomorphism* if $f(x + y) \subseteq f(x) + f(y)$ and $f(r \cdot x) \subseteq r \cdot f(x)$ for all $x, y \in M$ and $r \in R$. Finally, f is called a *strong homomorphism* if for all $x, y \in M$ and $x \in R$, we have $f(x + y) = f(x) + f(y)$ and $f(r \cdot x) = r \cdot f(x)$.

Lemma 3.5 ([3]). *Let M_1 and M_2 be two H_v-modules over an H_v-ring R and $f : M_1 \to M_2$ a strong epimorphism. If N is an H_v-submodule of M_2, then $f^{-1}(N)$ is an H_v-submodule of M_1.*

Theorem 3.6. *Let M_1 and M_2 be two H_v-modules, f a strong homomorphism from H_1 into H_2 and T (resp. S) an idempotent interval t-norm (resp. s-norm).*

(i) *If $A = (\widetilde{M}_A, \widetilde{N}_A)$ is an interval valued intuitionistic (S,T)-fuzzy H_v-submodule of M_1, then the image $f[A]$ of A is an interval intuitionistic (S,T)-fuzzy H_v-submodule of M_2.*

(ii) *If $B = (\widetilde{M}_B, \widetilde{N}_B)$ be an interval valued intuitionistic (S,T)-fuzzy H_v-submodule of M_2, then the inverse image $f^{-1}(B)$ of B is an interval valued intuitionistic (S,T)-fuzzy H_v-submodule of M_1.*

Proof. (i) Let $A = (\widetilde{M}_A, \widetilde{N}_A)$ be an interval valued intuitionistic (S,T)-fuzzy H_v-submodule of M_1. By Theorem 3.2, $U(\widetilde{M}_A;[t,s])$ and $L(\widetilde{N}_A;[t,s])$ are H_v-submodules of M_1 for every $[t,s] \in D[0,1]$. Therefore, by Lemma 3.5, $f(U(\widetilde{M}_A;[t,s])$ and $f(L(\widetilde{N}_A;[t,s]))$ are H_v-submodules of M_2. But $U(f(\widetilde{M}_A);[t,s]) = f(U(\widetilde{M}_A;[t,s]))$ and $L(f(\widetilde{N}_A);[t,s]) = f(L(\widetilde{N}_A;[t,s]))$, so $U(f(\widetilde{M}_A);[t,s])$ and $L(f(\widetilde{N}_A);[t,s])$ are H_v-submodules of M_2. Therefore $f[A]$ is an interval valued intuitionistic (S,T)-fuzzy H_v-submodule of M_2.

(ii) For any $x, y \in H$ and $\alpha \in x+y$, we have

$$\widetilde{M}_{f^{-1}(B)}(\alpha) = \widetilde{M}_B(f(\alpha)) \geq T(\widetilde{M}_B(f(x)), \widetilde{M}_B(f(y))) = T(\widetilde{M}_{f^{-1}(B)}(x), \widetilde{M}_{f^{-1}(B)}(y)).$$

Therefore

$$\inf_{\alpha \in x+y} \widetilde{M}_{f^{-1}(B)}(\alpha) \geq T(\widetilde{M}_{f^{-1}(B)}(x), \widetilde{M}_{f^{-1}(B)}(y)).$$

For $x, a \in M_2$ there exists $y \in M_2$ such that $x \in a+y$. Thus $f(x) \in f(a)+f(y)$ and

$$T(\widetilde{M}_{f^{-1}(B)}(x), \widetilde{M}_{f^{-1}(B)}(a)) = T(\widetilde{M_B}(f(x)), \widetilde{M_B}(f(a))) \leq \widetilde{M_B}(f(y)) = \widetilde{M}_{f^{-1}(B)}(y).$$

In the same manner, we can show that for $x, a \in M_2$ there exists $z \in M_2$ such that $x \in z+a$ and $T(\widetilde{M}_{f^{-1}(B)}(x), \widetilde{M}_{f^{-1}(B)}(a)) \leq \widetilde{M}_{f^{-1}(B)}(z)$.

It is not difficult to see that, for all $x \in M_2$, $r \in R$ and $\alpha \in r \cdot x$, we have $\widetilde{M}_{f^{-1}(B)}(\alpha) = \widetilde{M}(f^{-1}(\alpha)) \geq \widetilde{M}(f(x)) = \widetilde{M}_{f^{-1}(B)}(x)$, whence $\inf\limits_{\alpha \in r \cdot x} \widetilde{M}_{f^{-1}(B)}(\alpha) \geq \widetilde{M}_{f^{-1}(B)}(x)$. This completes the proof that $\widetilde{M}_{f^{-1}(B)}$ is an interval valued T-submodule of M_1.

Similarly, we can prove $\widetilde{N}_{f^{-1}(B)}$ is an interval valued S-fuzzy H_v-submodule of M_1. Therefore $f^{-1}(B)$ is an interval valued intuitionistic (S,T)-fuzzy H_v-submodule of M_1. $\square$

The mail tools in the theory of H_v-structures are the fundamental relations. Consider an H_v-module M over an H_v-ring R. If the relation γ^* is the smallest equivalence relation on R such that the quotient R/γ^* is a ring, we say that γ^* is the *fundamental equivalence relation* on R and R/γ^* is the *fundamental ring*. The fundamental relation ε^* on M over R is the smallest equivalence relation on M such that M/ε^* is a module over the ring R/γ^* (see [9, 10]).

Let $\mathcal{U}$ be the set of all expressions consisting of finite hyperoperations of either on R and M or the external hyperoperation applied on finite sets of R and M. Then a relation ε can be defined on M whose transitive closure is the fundamental relation ε^*. The relation ε is as follows:

$$x \varepsilon y \Longleftrightarrow \{x, y\} \subseteq u \quad \text{for some} \quad u \in \mathcal{U}.$$

Let us denote $\widetilde{\varepsilon}$ the transitive closure of ε. Then we can rewrite the definition of $\widetilde{\varepsilon}$ on M as follows:

$$a\widetilde{\varepsilon}b \Longleftrightarrow \begin{cases} \text{there exist } z_1, z_2, \ldots, z_{n+1} \in M \text{ and } u_1, u_2, \ldots, u_n \in \mathcal{U} \\ \text{such that } z_1 = a,\ z_{n+1} = b, \text{ and } \{z_i, z_{i+1}\} \subseteq u_i \text{ for all } i = 1, \ldots, n. \end{cases}$$

The fundamental relation ε^* is the transitive closure of the relation ε (see [11]).

Suppose $\gamma^*(r)$ is the equivalence class containing $r \in R$ and $\varepsilon^*(x)$ is the equivalence class containing $x \in M$. On M/ε^*, the sum $\oplus$ and the external product $\odot$ using the γ^* classes in R, are defined as follows:

$$\varepsilon^*(x) \oplus \varepsilon^*(y) = \varepsilon^*(c) \quad \text{for all } c \in \varepsilon^*(x) + \varepsilon^*(y),$$

$$\gamma^*(r) \odot \varepsilon^*(x) = \varepsilon^*(d) \quad \text{for all } d \in \gamma^*(r) \cdot \varepsilon^*(x).$$

The kernel of the canonical map $\varphi : M \to M/\varepsilon^*$ is called the *core* of M and is denoted by ω_M. Here we also denote by ω_M the zero element of the group $(M/\varepsilon^*, \oplus)$. Also, we have $\omega_M = \varepsilon^*(0)$ and $\varepsilon^*(-x) = -\varepsilon^*(x)$ for all $x \in M$.

Definition 3.7. Let $A = (\widetilde{M}_A, \widetilde{N}_A)$ be an interval valued intuitionistic fuzzy set. The intuitionistic fuzzy set $A/\varepsilon^* = (\widetilde{M}_{\varepsilon*}, \widetilde{N}_{\varepsilon*})$ is defined as the pair of maps

$$\begin{cases} \widetilde{M}_{\varepsilon*} : M/\varepsilon^* \to D[0,1], \\ \widetilde{N}_{\varepsilon*} : N/\varepsilon^* \to D[0,1] \end{cases}$$

such that

$$\widetilde{M}_{\varepsilon^*}(\varepsilon^*(x)) = \begin{cases} \sup\limits_{a \in \varepsilon^*(x)} \widetilde{M}_A(a) & \text{if } \varepsilon^*(x) \neq \omega_M \\ [1,1] & \text{otherwise} \end{cases}$$

and

$$\widetilde{N}_{\varepsilon^*}(\varepsilon^*(x)) = \begin{cases} \inf\limits_{a \in \varepsilon^*(x)} \widetilde{N}_A(a) & \text{if } \varepsilon^*(x) \neq \omega_M \\ [0,0] & \text{otherwise.} \end{cases}$$

Definition 3.8. Let T (resp. S) be an idempotent interval t-norm (resp. s-norm). An interval valued intuitionistic fuzzy set $A = (\widetilde{M}_A, \widetilde{N}_A)$ on an ordinary module M over a ring R is called an *interval valued intuitionistic* (S,T)*-fuzzy submodule* of M, if

(*i*) $\widetilde{M}_A(0) = [1,1]$ and $\widetilde{N}_A(0) = [0,0]$,

(*ii*) $T(\widetilde{M}_A(x), \widetilde{M}_A(y)) \leq \widetilde{M}_A(x-y)$ and $S(\widetilde{N}_A(x), \widetilde{N}_A(y)) \geq \widetilde{N}_A(x-y)$ for all $x, y \in M$,

(*iii*) $\widetilde{M}_A(x) \leq \widetilde{M}_A(r \cdot x)$ and $\widetilde{N}_A(x) \geq \widetilde{N}_A(r \cdot x)$ for all $x \in M$ and $r \in R$.

Theorem 3.9. *Let M be an H_v-submodule of M over an H_v-ring. If $A = (\widetilde{M}_A, \widetilde{N}_A)$ is an interval valued intuitionistic (S,T)-fuzzy H_v-submodule of M, then A/ε^* is an interval valued intuitionistic (S,T)-fuzzy submodule of the fundamental module M/ε^*.*

Proof. The first condition of the above definition is trivially satisfied. To prove the second consider two arbitrary elements $\varepsilon^*(x)$, $\varepsilon^*(y)$ of M/ε^*.

If $\varepsilon^*(x) = \omega_M$, then

$$T(\widetilde{M}_{\varepsilon^*}(\varepsilon^*(x)), \widetilde{M}_{\varepsilon^*}(\varepsilon^*(y))\} = T([1,1], \widetilde{M}_{\varepsilon^*}(\varepsilon^*(y))) = \widetilde{M}_{\varepsilon^*}(\varepsilon^*(y)) = \widetilde{M}_{\varepsilon^*}(\varepsilon^*(x) \oplus \varepsilon^*(y)).$$

If $\varepsilon^*(x) \neq \omega_M$, then

$$\begin{aligned}
T(\widetilde{M}_{\varepsilon^*}(\varepsilon^*(x)), \widetilde{M}_{\varepsilon^*}(\varepsilon^*(y))) &= T(\sup_{a\in\varepsilon^*(x)} \widetilde{M}_A(a), \sup_{b\in\varepsilon^*(y)} \widetilde{M}_A(b)) = \sup_{\substack{a\in\varepsilon^*(x)\\ b\in\varepsilon^*(y)}} T(\widetilde{M}_A(a), \widetilde{M}_A(b)) \\
&\leq \sup_{\substack{a\in\varepsilon^*(x)\\ b\in\varepsilon^*(y)}} \big(\inf_{\alpha\in a+b} \widetilde{M}_A(\alpha)\big) \leq \sup_{\substack{a\in\varepsilon^*(x)\\ b\in\varepsilon^*(y)}} \big(\sup_{\alpha\in a+b} \widetilde{M}_A(\alpha)\big) \\
&\leq \sup_{\substack{a\in\varepsilon^*(x)\\ b\in\varepsilon^*(y)}} \big(\sup_{\alpha\in\varepsilon^*(a+b)} \widetilde{M}_A(\alpha)\big) = \sup_{\substack{a\in\varepsilon^*(x)\\ b\in\varepsilon^*(y)}} \big(\widetilde{M}_{\varepsilon^*}(\varepsilon^*(a+b))\big) \\
&= \widetilde{M}_{\varepsilon^*}(\varepsilon^*(a+b))
\end{aligned}$$

for all $a \in \varepsilon^*(x)$ and $b \in \varepsilon^*(y)$. Hence

$$\widetilde{M}_{\varepsilon^*}(\varepsilon^*(a+b)) = \widetilde{M}_{\varepsilon^*}(\varepsilon^*(x+y)) = \widetilde{M}_{\varepsilon^*}(\varepsilon^*(x) \oplus \varepsilon^*(y)).$$

So,

$$T(\widetilde{M}_{\varepsilon^*}(\varepsilon^*(x)), \widetilde{M}_{\varepsilon^*}(\varepsilon^*(y))) \leq \widetilde{M}_{\varepsilon^*}(\varepsilon^*(x) \oplus \varepsilon^*(y)). \tag{1}$$

The proof of the inequality

$$S(\widetilde{N}_{\varepsilon^*}(\varepsilon^*(x)), \widetilde{N}_{\varepsilon^*}(\varepsilon^*(y))) \geq \widetilde{N}_{\varepsilon^*}(\varepsilon^*(x) \oplus \varepsilon^*(y)) \tag{2}$$

is similar.

Let $\varepsilon^*(x)$ and $\varepsilon^*(a)$ be two arbitrary elements of M/ε^*. Because $A = (\widetilde{M}_A, \widetilde{N}_A)$ is an interval valued intuitionistic (S,T)-fuzzy H_v-submodule of M, then for every $t \in \varepsilon^*(a)$ and $s \in \varepsilon^*(x)$, there exists $y_{t,s} \in M$ such that $t \in s + y_{t,s}$ and $T(\widetilde{M}_A(t), \widetilde{M}_A(s)) \leq \widetilde{M}_A(y_{t,s})$. From $t \in s + y_{t,s}$, it follows that $\varepsilon^*(s) \oplus \varepsilon^*(y_{t,s}) = \varepsilon^*(t)$, i.e. $\varepsilon^*(x) \oplus \varepsilon^*(y_{t,s}) = \varepsilon^*(a)$.

If $\varepsilon^*(a) \neq \omega_M$, $\varepsilon^*(x) \neq \omega_M$, then

$$\begin{aligned}
T(\widetilde{M}_{\varepsilon^*}(\varepsilon^*(a)), \widetilde{M}_{\varepsilon^*}(\varepsilon^*(x))) &= T(\sup_{t\in\varepsilon^*(a)} \widetilde{M}_A(t), \sup_{s\in\varepsilon^*(a)} \widetilde{M}_A(s)) = \sup_{\substack{t\in\varepsilon^*(a)\\ s\in\varepsilon^*(x)}} T(\widetilde{M}_A(t), \widetilde{M}_A(s)) \\
&\leq \sup_{\substack{t\in\varepsilon^*(a)\\ s\in\varepsilon^*(x)}} \widetilde{M}_A(y_{t,s}) \leq \sup_{y\in\varepsilon^*(y_{t,s})} \widetilde{M}_A(y) = \widetilde{M}_{\varepsilon^*}(\varepsilon^*(y_{t,s})),
\end{aligned}$$

i.e.

$$T(\widetilde{M}_{\varepsilon^*}(\varepsilon^*(a)), \widetilde{M}_{\varepsilon^*}(\varepsilon^*(x))) \leq \widetilde{M}_{\varepsilon^*}(\varepsilon^*(y_{t,s})).$$

Similarly

$$S(\widetilde{N}_{\varepsilon^*}(\varepsilon^*(a)), \widetilde{N}_{\varepsilon^*}(\varepsilon^*(x))) \geq \widetilde{N}_{\varepsilon^*}(\varepsilon^*(y_{t,s})).$$

If $\varepsilon^*(x) = \omega_M(x)$, then $\varepsilon^*(a) = \varepsilon^*(y_{t,s})$. So,

$$T(\widetilde{M}_{\varepsilon^*}(\varepsilon^*(a)), \widetilde{M}_{\varepsilon^*}(\varepsilon^*(x))) \leq \widetilde{M}_{\varepsilon^*}(\varepsilon^*(a)) = \widetilde{M}_{\varepsilon^*}(\varepsilon^*(y_{t,s}))$$

and

$$S(\widetilde{N}_{\varepsilon^*}(\varepsilon^*(a)), \widetilde{N}_{\varepsilon^*}(\varepsilon^*(x))) \geq \widetilde{N}_{\varepsilon^*}(\varepsilon^*(a)) = \widetilde{N}_{\varepsilon^*}(\varepsilon^*(y_{t,s})).$$

Therefore for all $\varepsilon^*(x), \varepsilon^*(a) \in M/\varepsilon^*$, there exists $\varepsilon^*(y) \in M/\varepsilon^*$ such that $\varepsilon^*(x) = \varepsilon^*(a) \oplus \varepsilon^*(y)$ for which

$$T(\widetilde{M}_{\varepsilon^*}(\varepsilon^*(x)), \widetilde{M}_{\varepsilon^*}(\varepsilon^*(a))) \leq \widetilde{M}_{\varepsilon^*}(\varepsilon^*(y))$$

and

$$S(\widetilde{N}_{\varepsilon^*}(\varepsilon^*(x)), \widetilde{N}_{\varepsilon^*}(\varepsilon^*(a))) \geq \widetilde{N}_{\varepsilon^*}(\varepsilon^*(y)).$$

From the above it follows that for all $\varepsilon^*(x) \in M/\varepsilon^*$ we have $\widetilde{M}_{\varepsilon^*}(\varepsilon^*(x)) \leq \widetilde{M}_{\varepsilon^*}(-\varepsilon^*(x))$ and $\widetilde{N}_{\varepsilon^*}(\varepsilon^*(x)) \geq \widetilde{N}_{\varepsilon^*}(-\varepsilon^*(x))$. Indeed, for $\omega_M \in M/\varepsilon^*$ there exists $\varepsilon^*(y_1) \in M/\varepsilon^*$ such that $\omega_M = \varepsilon^*(x) \oplus \varepsilon^*(y_1)$ and

$$\widetilde{M}_{\varepsilon^*}(\varepsilon^*(x)) = T([1,1], \widetilde{M}_{\varepsilon^*}(\varepsilon^*(x)) = T(\widetilde{M}_{\varepsilon^*}(\omega_M), \widetilde{M}_{\varepsilon^*}(\varepsilon^*(x))) \leq \widetilde{M}_{\varepsilon^*}(\varepsilon^*(y_1)),$$

$$\widetilde{N}_{\varepsilon^*}(\varepsilon^*(x)) = S([0,0], \widetilde{N}_{\varepsilon^*}(\varepsilon^*(x)) = S(\widetilde{N}_{\varepsilon^*}(\omega_M), \widetilde{N}_{\varepsilon^*}(\varepsilon^*(x))) \geq \widetilde{N}_{\varepsilon^*}(\varepsilon^*(y_1)),$$

because $\widetilde{M}_{\varepsilon^*}(\omega_M) = [1,1]$, $\widetilde{N}_{\varepsilon^*}(\omega_M) = [0,0]$. But $\omega_M = \varepsilon^*(x) \oplus \varepsilon^*(y_1)$ implies $\varepsilon^*(y_1) = -\varepsilon^*(x)$. Therefore

$$\widetilde{M}_{\varepsilon^*}(\varepsilon^*(x)) \leq \widetilde{M}_{\varepsilon^*}(-\varepsilon^*(x)), \qquad \widetilde{N}_{\varepsilon^*}(\varepsilon^*(x)) \geq \widetilde{N}_{\varepsilon^*}(-\varepsilon^*(x)). \tag{3}$$

So, by (1), (2) and (3), for all $\varepsilon^*(x), \varepsilon^*(y) \in M/\varepsilon^*$, we have

$$\begin{aligned}\widetilde{M}_{\varepsilon^*}(\varepsilon^*(x) - \varepsilon^*(y)) &= \widetilde{M}_{\varepsilon^*}(\varepsilon^*(x) \oplus (-\varepsilon^*(y))) \geq T(\widetilde{M}_{\varepsilon^*}(\varepsilon^*(x)), \widetilde{M}_{\varepsilon^*}(-\varepsilon^*(y))) \\ &\geq T(\widetilde{M}_{\varepsilon^*}(\varepsilon^*(x)), \widetilde{M}_{\varepsilon^*}(\varepsilon^*(y)))\end{aligned}$$

and

$$\begin{aligned}\widetilde{N}_{\varepsilon^*}(\varepsilon^*(x) - \varepsilon^*(y)) &= \widetilde{N}_{\varepsilon^*}(\varepsilon^*(x) \oplus (-\varepsilon^*(y))) \leq S(\widetilde{N}_{\varepsilon^*}(\varepsilon^*(x)), \widetilde{N}_{\varepsilon^*}(-\varepsilon^*(y))) \\ &\leq S(\widetilde{N}_{\varepsilon^*}(\varepsilon^*(x)), \widetilde{N}_{\varepsilon^*}(\varepsilon^*(y))).\end{aligned}$$

This completes the proof of the second condition of Definition 3.8.

To prove the third condition observe that if $\varepsilon^*(x) \in M/\varepsilon^*$ and $\gamma^*(r) \in R/\gamma^*$, then $\widetilde{M}_{\varepsilon^*}(\gamma^*(r) \odot \varepsilon^*(x)) = \widetilde{M}_{\varepsilon^*}(\varepsilon^*(r \cdot b))$ for every $b \in \varepsilon^*(x)$ and

$$\widetilde{M}_{\varepsilon^*}(\varepsilon^*(r \cdot b)) = \sup_{\alpha \in \varepsilon^*(r \cdot b)} \widetilde{M}_A(\alpha) \geq \sup_{\alpha \in r \cdot b} \widetilde{M}_A(\alpha) \geq \widetilde{M}_A(b).$$

Hence $\widetilde{M}_{\varepsilon^*}(\gamma^*(r) \odot \varepsilon^*(x)) \geq \sup\limits_{b \in \varepsilon^*(x)} \widetilde{M}_A(b)$, which implies $\widetilde{M}_{\varepsilon^*}(\gamma^*(r) \odot \varepsilon^*(x)) \geq \widetilde{M}_{\varepsilon^*}(\varepsilon^*(x))$.

Similarly, we obtain $\widetilde{N}_{\varepsilon^*}(\gamma^*(r) \odot \varepsilon^*(x)) \leq \widetilde{N}_{\varepsilon^*}(\varepsilon^*(x))$. This completes the proof. □

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